Hard Constraints Explicit Model Predictive Control of an Inverted Pendulum

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***Abstract*— In this paper, explicit model predictive controller is applied to an inverted pendulum apparatus. Explicit solutions to constrained linear model predictive controller can be computed by solving multi-parametric quadratic programs. The solution is a piecewise affine function, which can be evaluated at each sample to obtain the optimal control law. The on-line computation effort is restricted to a table-lookup. This admits implementation on low cost hardware at high sampling frequencies in real-time systems with high reliability and low software complexity. This is useful for systems with limited power and CPU resources.**

Keywords-component; explicit model predictive control, inverted pendulum, constrained systems

I. Introduction

The Inverted Pendulum is a classical example of how the use of control may be employed to stabilize an inherently unstable system. The Inverted pendulum system represents also an accurate model for pitch and yaw behaviors of a flying rocket and can be used as a benchmark for many control methodologies. The Segway PT is a two wheeled (in parallel), self-balancing vehicle that transports a single person which uses the properties of the inverted pendulum, [1].

Model predictive control (MPC), also referred to as moving horizon control or receding horizon control, has become an attractive feedback strategy, especially for linear or nonlinear systems subject to input and state constraints. In general, linear and nonlinear MPC are distinguished. Linear MPC refers to a family of MPC schemes in which linear models are used to predict the system dynamics, even though the dynamics of the closed loop system is nonlinear due to the presence of constraints. Linear MPC approaches have found successful applications, especially in the process industries

The success of MPC is due to the fact that it is perhaps the most general way of posing the control problem in the time domain. The use of a finite-horizon strategy allows the explicit handling of process and operational constraints by the MPC, [2].

The strengths of MPC lie mainly in its handling of constraints and its relative simplicity. Commonly, a quadratic performance measure over a finite prediction horizon is employed. Problems of this type are easily translated into a quadratic program.

Standard predictive control involves solving the optimization problem at every sampling instant, based on the value of the current state vector. For this reason, MPC has been traditionally labeled as a technology for slow processes. The complexity still is prohibitive for fast plants and low-end embedded computers. Microcontroller and computer technology are progressively advancing the computation speed, but still solving optimization problem on line prevents the application of MPC in many contexts. Another limitation is software certification issues. The code implementing the solver might generate concerns due to software certification issues, a problem which is particularly acute in safety critical applications.

A remedy for these limitations is to use explicit model predictive control. In [3, 4] it was recognized that the constrained linear MPC problem can be posed as a multi-parametric quadratic program (mp-QP), when the state is viewed as a parameter to the problem. It was shown that the control input, which is the solution of the mp-QP, has an explicit representation as a piecewise linear state feedback on a polyhedral partition of the state space. The mp-QP algorithm is developed to compute this function, [5-7]. For a given range of operating condition, explicit MPC solves the optimization problem off-line. By exploiting multi-parametric programming techniques, explicit MPC computes the optimal control action off-line as an explicit function of the state vector. Such a function is piecewise affine, so that the MPC maps into a lookup table of linear gains.

Disturbance rejection is another topic in predictive control that requires special consideration in low-level control applications, [8].

This research is funded in part by Ministry of Science, Technology and Innovation (MOSTI) under the e-Science fund project no. 06-02-12-SF0072.

This paper presents an application of an explicit model predictive controller on an inverted pendulum, a process that is unstable and non-minimum phase.

The outline of the paper is as follows: The following section describes the explicit model predictive controller. Section III presents the inverted pendulum process. Section IV details tuning procedure and performance of the closed loop system. Section V provides experimental results.

II. Explicit Model Predictive Controller

In this paper, the process to be controlled can be described by a discrete-time, deterministic linear state space model, that is

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$

$$y(k) = Cx(k) \quad (2)$$

where $x(k) \in \Re^n$, $u(k) \in \Re^m$ and $y(k) \in \Re^p$ are the state, input and output variable. Also, $A \in \Re^{n\times m}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and (A, B) is a controllable pair. It is assumed that a full measurement of the state $x(k)$ is available at the current time *k*.

If we now consider the regulator problem, that is, the problem of driving the state vector to the origin, the traditional MPC solves the following optimization problem for the current $x(k)$:

$$V^*(x(k)) = \min_U \{J(U, x(k))\} \quad (3)$$

subject to:

$$\begin{aligned}
&x_{\min} \le x_{k+i|k} \le x_{\max},\ i = 1,\cdots,N_p - 1\\
&u_{\min} \le u_{k+i|k} \le u_{\max},\ i = 1,\cdots,N_u - 1\\
&x_{k+N|k} \in \chi_N\\
&x_{k|k} = x(k)\\
&x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i},\ i \ge 0\\
&y_{k+i|k} = Cx_{k+i|k},\ i \ge 0\\
&u_{k+i} = Kx_{k+i|k},\ N_u \le i \le N_p
\end{aligned} \quad (4)$$

with the cost function given by:

$$J(U, x(k)) := x_{k+N|k}^T P x_{k+N|k} + \sum_{i=0}^{N_p - 1} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i}^T R u_{k+i} \quad (5)$$

The column vector $U := \left[u_k^T, \cdots, u_{k+N_u-1}^T\right]^T$, $U \in \Re^\ell$, $\ell = mN_u$ is the optimization vector and χ_N is polyhedral. $x_{k+i|k}$ denotes the predicted state vector at time $k+i$, obtained by applying the input sequence $u_k, \cdots, u_{k+i-1}$ to model described by (1) and (2) starting from the state $x(k)$. N_p is prediction horizon, and N_u is control horizon, and *Q*, *P* and *R* are square, symmetric and positive definite matrices.

The final cost matrix *P* may be taken as the solution of the discrete-time algebraic Riccati equation:

$$P = A^T PA - A^T PB\left(B_T PB + R\right)^{-1}\left(A^T PB\right)^T + Q \quad (6)$$

With the assumption that no constraints are active for $k \ge N$, this corresponds to an infinite horizon LQ criterion, and the MPC is stabilizing.

By substituting

$$x_{k+i|k} = A^i x(k) + \sum_{j=0}^{i-1} A^i B u_{i-1-j} \quad (7)$$

problem (3) becomes a quadratic program:

$$J^*(U, x(k)) = \frac{1}{2} x^T(k) Y x(k) + \min_U \left\{\frac{1}{2} U^T HU + x^T(k) FU\right\} \quad (8)$$

subject to:

$$GU \le W + Ex(k) \quad (9)$$

where *H, F, Y, G, W, E* are easily obtained from *Q, R,* and (3)–(5). Assuming that $H > 0$ the problem is strictly convex, and the Karush-Kuhn-Tucker conditions (KKT) are sufficient conditions for optimality, giving a unique solution U^* for (3). By ensuring that *Q* and *R* are positive semi-definite and positive definite, respectively, the assumption $H > 0$ is indeed satisfied.

Further, by defining $z := U + H^{-1}Fx(k)$, $z \in \Re^s$, $s = mN_u$, the optimization problem (3) is transformed into the following equivalent problem:

$$V_z^*(x(k)) = \min_z \left\{\frac{1}{2} z^T Hz\right\} \quad (10)$$

subject to:

$$Gz \le W + Sx(k) \tag{11}$$

where $V_z^*(x(k)) := V^*(x(k)) - \frac{1}{2}x(k)^T\left(Y - FH^{-1}F^T\right)x(k)$, $S := E + GH^{-1}F^T$, $H \in \Re^{s\times s}$, $G \in \Re^{q\times s}$, $W \in \Re^{q\times 1}$, $S \in \Re^{q\times n}$ and q is the number of inequalities. The vector $x(k)$ is the current state, which can be treated as a vector of parameters.

It has been shown in [3, 4] that the optimization problem (10) is a multi-parametric quadratic program (mp-QP) and its solution can be found in an explicit form $z^* = z^*(x(k))$ as a PWL function of $x(k)$ defined over a polyhedral partition of the parameter space. Algorithms for iteratively constructing a polyhedral partition of the state space and computing the PWL solution are given in [4-7].

III. Process Description

The rotary pendulum module consists of a flat arm which is instrumented with a sensor at one end such that the sensor shaft is aligned with the longitudinal axis of the arm. A fixture is supplied to attach the pendulum to the sensor shaft. The opposite end of the arm is designed to be mounted on a rotary servo plant resulting in a horizontally rotating arm with a pendulum at the end.

The system is identified and linearized around the origin. As the internal model, the following model is used.

$$x(k+1) = Ax(k) + Bu(k) \tag{12}$$

$$y(k) = Cx(k) \tag{13}$$

where

$$A = \begin{bmatrix} 1 & 0.001876 & 0.009308 & 6.3\times10^{-6} \\ 0 & 1 & -0.00066 & 0.01 \\ 0 & 0.3665 & 0.8647 & 0.001876 \\ 0 & 0.7927 & -0.1303 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.001218 \\ 0.001173 \\ 0.2379 \\ 0.2292 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad x^T = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}$$

State θ is arm angle and α is pendulum angle. A sampling period T_s of 10ms is used. The open loop system is unstable.

IV. Real-Time Performance of Explicit MPC

In the remainder of this paper we consider the application of MPC to an inverted pendulum apparatus shown in Fig. 1.

A. Constraints

There are physical limits on the control input and the arm position, which correspond to constraints on the supply current to the motor and the angle of the arm, respectively.

The input to the motor is constrained to lie between $-12 \le u(k) \le 12$ (volts) and the arm position must lie between $-1 \le \theta \le 1$ (rad).

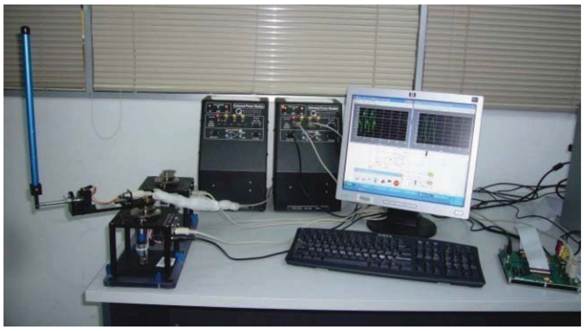

Figure 1. Inverted pendulum and controller.

B. Design of controller

We would like the state $x(k)$ to be at the origin, which corresponds to the arm position at the null position, the pendulum angle of 0 radians (i.e. upright), the arm not moving and the pendulum not rotating. Furthermore, we would like to be economical with control action and thus penalize input movements. This objective can be described in terms of the cost function in Eq. (9) via the choices

$$Q = diag(q_1, q_2, q_3, q_4), \; R = r \tag{14}$$

Since it is important that the pendulum angle is zero, q_2 receives a high value. It is less important, but not insignificant, that the arm position is zero, so q_1 has the next highest value. Arm velocity and pendulum angular velocity are not so important, so q_3 and q_4 receive zero value. For the experimental results shown in Section V this corresponds to

$$q_1 = 1,\; q_2 = 5,\; q_3 = 0,\; q_4 = 0,\; r = 0.1 \tag{15}$$

Both prediction horizon N_p and control horizon N_u have been established based on the assumptions that large values lead to increased computational effort and short values produce short-sighted control policy. The value of

$$N_p = 50 \tag{16}$$

was selected for the prediction horizon. For the control horizon the value of

$$N_u = 4 \tag{17}$$

has been taken. The control horizon is established not too long, to prevent computational complexity, and also not too short, to prevent an inefficient control and to provide a sufficient number of degrees of freedom.

V. RESULTS

The experimental results from applying the explicit model predictive controller described above to the inverted pendulum shown in Fig. 1 are provided in this section. The quadratic programming is used to solve the optimization problem. All plots in this section show data recorded from the physical apparatus by the PC and data acquisition hardware.

The initial position of the arm is in the null position and the pendulum tip is down (in the stable position). First the swing up controller is switched on and changes the position of the pendulum to upright position [9, 10]. Then the explicit model predictive controller is switched on and keeps the pendulum upward and rejects the disturbances.

To gauge the utility of the explicit model predictive controller, a large disturbance was manually applied to the pendulum tip while it was in the upright position. Fig. 2 shows the response of pendulum angle to this disturbance. The response of arm angle is depicted in Fig. 3. The output signal of the explicit model predictive controller which is applied to DC motor is depicted in Fig. 4.

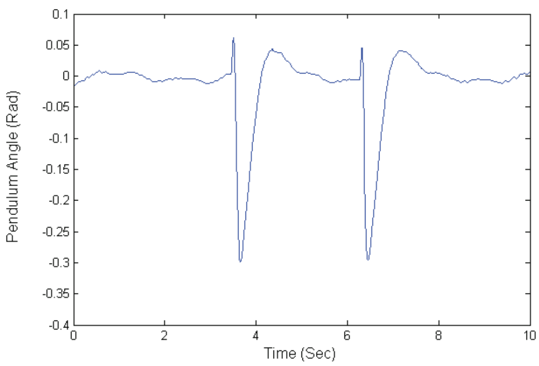

Figure 2. Pendulum angle.

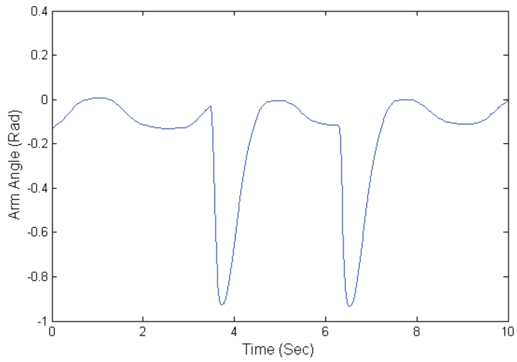

Figure 3. Arm angle.

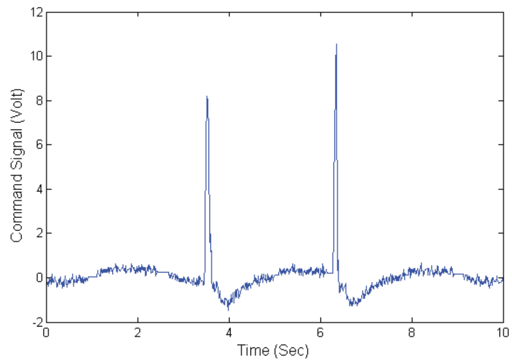

Figure 4. Output signal of the explicit model predictive controller which is applied to DC motor.

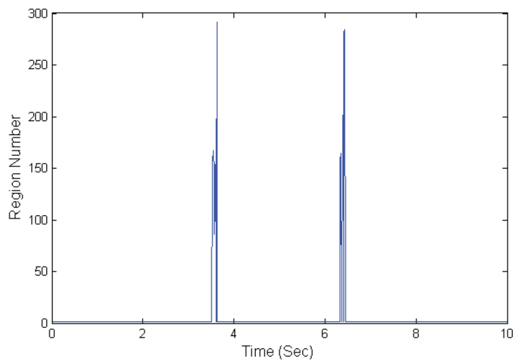

Figure 5. Regions.

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Note that the arm position and input obey their respective limits. If the disturbance amplitude is increased, the solution of the multi parametric quadratic program may be infeasible. This is the most important limitation of hard constraint model predictive control. A classical solution to this problem is soft constraint profile, which is out of the scope of this paper. Note also that friction is the most important nonlinear dynamic of the system which creates the limit cycle [11]. Fig. 2 and Fig. 3 show that there is stable oscillation around the equilibrium point.

VI. CONCLUSION

This paper presents the application of explicit model predictive control to an inverted pendulum apparatus. While the good performance of MPC for this application may be of independent interest, the key point is that a reasonably challenging control problem can be dealt with via MPC in realtime on a modest hardware platform at a 100Hz sample rate. MPC can reject the manual disturbance of the pendulum angle. Friction is the most important nonlinear dynamic of the system which creates the limit cycle.

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